On Optimum Finite Element Grid Configurations

David Turcke*
Queen's University, Kingston, Canada

Introduction

ONLY recently have the significant benefits accrued from optimum finite element mesh configurations been reported.[1-6] The criterion for this Note, and for the majority of the above research is concerned with finding a mesh which is absolutely better, in terms of the total potential energy Π_p, than an assumed initial grid topology. That is, to

$$\underset{(\{\Delta\},\{x\})}{\text{minimize}} \sum_{e}^{N} \Pi_p^e(\{\Delta\},\{c\}) \tag{1}$$

where $\sum_e$ is the notation for the summation over all N elements, $\{\Delta\}$ is a set of nodal displacements, and $\{x\}$ is a set of nodal coordinates.

The procedures developed by the author avoid the explicit mathematical operations involved with Eq. (1) and constitute the first systematic method for establishing near optimum grid layouts. These guidelines have 3 fundamental aspects. Firstly, to obtain a plot of the isoenergetics which are contours of constant strain energy density, from an initial analysis, secondly, to determine the near optimum position of selected contours and consequent element gradation, and thirdly, to lay out the grid. The purpose of this Note is to present a more rational technique in step 2 of the guidelines for locating the isoenergetics, or grading the finite elements in high strain gradient regions.

Present Grading Scheme

At present the determination of the location of the isoenergetic contours has been based simply on using straight line segments to piecewise approximate the the strain energy density variation along lines of high energy density gradient. This variation is determined from an initial analysis. The intersection or 'join' points of the linear segments on each line provide the locations through which the isoenergetics can be drawn. Thus a gradation of the elements away from the high gradient region is obtained such that an improved approximation can be achieved in subsequent analyses. This procedure requires the analyst to employ his own criterion for the previous approximation. In addition the use of straight line segments to approximate the strain energy density variation is inconsistent with the linear displacement models used in the previous studies.[3,4]

Proposed Grading Scheme

An investigation was carried out[1,6] for various one-dimensional problems, with and without strain singularities and necessarily possessing finite energy. It was demonstrated numerically and proven theoretically that the average strain at the optimum nodal locations was exact even though only an approximate displacement solution was obtained. It is precisely this exactness proof, previously mentioned that has led to the proposed grading scheme.

That is, the location of the isoenergetic contours can be arrived at by minimizing the following expression[7]

$$\underset{(\{\Delta\},\{x\})}{\text{minimize}} \sum_{e}^{N} \int_V (U^e - U_0^e)^2 dv \tag{2}$$

where U^e and U_0^e are the exact and approximate strain energy density expressions respectively within each element. Consequently this leads to the following criteria: 1) In each element the constant value of the approximating strain energy density U_0^e must be the average value of the exact function U^e within the element subdomain, and 2) On the boundary between elements the average value of the discontinuous strain energy density must be equal to the exact value U^e.

These criteria are nothing more than a generalization of the previously mentioned proof of exactness. However it is more rational than the present grading scheme. It provides an explicit grading technique along with piecewise constant value strain energy density approximations consistent with the linear displacement models studied. In addition, the explicit evaluation of Eq. (2) is avoided, and the analyst can employ the criteria directly, once an approximation to U^e is obtained. In the application of the criteria an initial analysis provides the analyst with an approximation to U^e, since the exact function U^e is obtained.

An investigation of a one-dimensional tapered bar problem with a variable number of nodes and tapers, *H1/H2* shown in Fig. 1 provided the following important result. Using Eq. (2) and incorporating the exact strain energy density function U^e, the resulting optimal nodal positions, as shown in Table 1, consistently overestimated the optimal nodal locations employing Eq. (1). That is the nodes have been drawn further into the high strain gradient region than by using Eq. (1). This is beneficial, since in the application of the guidelines, one generally obtains from an initial analysis with a coarse grid, a strain energy density function, which is used as an estimate of U^e in the criteria, having a lower gradient than the exact response function, see Fig. 2. Consequently this overestimation has proved very effective in modeling the actually higher strain energy density gradient, and arriving at near optimal nodal locations.

Application of Proposed Grading Scheme

The proposed procedure using Eq. (2) will be applied first to the one-dimensional problem in Fig. 1 having 4 elements ($N=4$) and a taper (*H2/H1*) equal to 6 and then to a two-dimensional problem shown in Fig. 3. In the one-dimensional example, an initial grid with equally spaced nodes, was used to obtain an approximate strain energy density variation U^e along the X axis as depicted in Fig. 2. The near optimum nodal locations were then found manually by adjusting U_0^e in a straightforward manner until the proposed high gradient regions with the optimal locations X_2, X_3, and X_4 indicated in Fig. 2 employing Eq. (1).

It should be remarked that once having established these new nodal locations, several re-analyses can be performed to obtain a new U^e along with re-applications of the grading scheme, until no significant change in the nodal locations or strain energy is noted. In this example the strain energy using the proposed grading scheme was within 0.21% of the energy using Eq. (1) explicitly.

Finally a two-dimensional plane stress problem shown in Fig. 3 is used to demonstrate the effectiveness of the criteria in a modified form. It has been shown[3,4,6] that in order to circumvent excessive computational efforts in the gradation or location of the isoenergetic contours for two-dimensional problems, an examination need only be made along trajectories having a high strain energy density gradient. This ef-

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*Assistant Professor of Civil Engineering

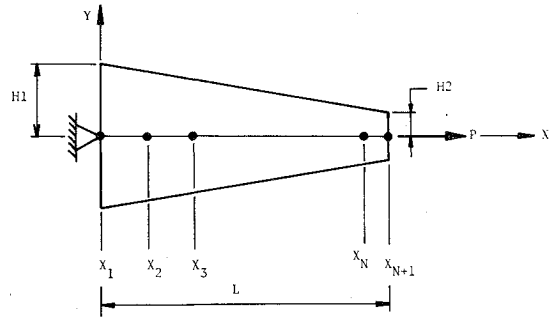

Fig. 1 General linear taper problem.

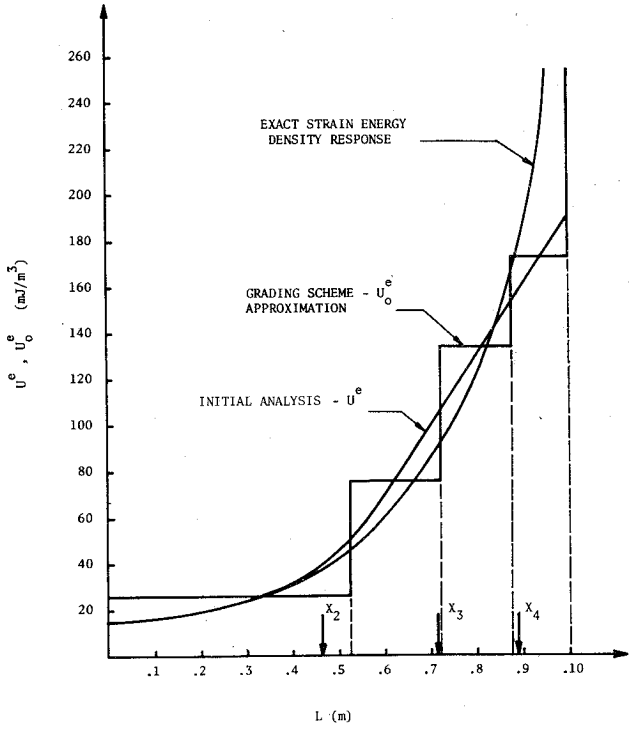

Fig. 2 One-dimensional application of proposed near optimum grading scheme.

fectively reduces the two-dimensional problem of contour positioning to a series of one-dimensional problems.

In general an initial analysis provides information with regard to the shape of the isoenergetics, establishes trajectories of high strain energy density gradient and permits a plot of U^e along each trajectory.

In this example, assuming that one is interested in locating only 3 contours, 4 piecewise segements must be approximated to U^e along the selected high strain energy density gradient lines AB and AC. Consequently this provides sufficient 'join' points on each such line to determine the position of the 3

Table 1 Optimal nodal positions

	Two Elements ($N=2$)		Three elements ($N=3$)			
Minimizing	$H1/H2$		$H1/H2$			
Expression	2/1	6/1	2/1		6/1	
	X_2/L	X_2/L	X_2/L	X_3/L	X_2/L	X_3/L
(1)	0.59	0.71	0.41	0.74	0.54	0.84
(2)	0.64	0.83	0.47	0.78	0.70	0.90

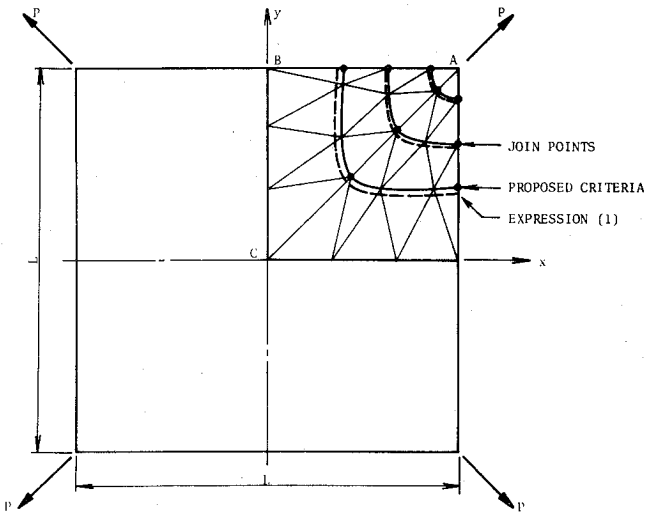

Fig. 3 Two-dimensional application of proposed near optimum grading scheme.

isoenergetics. Due to the regularity of the contour shape only these 2 trajectories need be used.

Applying the criteria along each trajectory, as was demonstrated for the one-dimensional example, the various 'join' points were determined as shown in Fig. 3. Subsequently the contour can be drawn in and then finally the mesh layed out according to the third step in the general mesh layout procedure.[3] As anticipated there is good agreement in the critical high gradient region between the proposed scheme and the optimum locations arrived at by using Eq. (1) explicitly. This is substantiated by the fact that the total strain energy using the proposed grading technique is within 0.5% of the strain energy using Eq. (1). In addition, not only is the proposed scheme more rational than the present procedure but is in better agreement in a strain energy sense than the present technique.[3]

Conclusions

A rational procedure for locating or grading finite elements has been presented, in order that the analyst can obtain near optimum mesh configurations in a straightforward and computationally inexpensive manner. The question of optimum grid layouts is a difficult one, however the resulting benefits[1-6] are sufficient to warrant further study using the proposed scheme.

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